Forward LIBOR model

Math 622

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1 Forward LIBOR

1.1 Continuous vs simple compounding

One "draw-back" of the forward rate, as we discussed, is that we cannot posit a log-normal distribution for $f(t,T)$ since that would cause the forward rate to explode near T (Shreve's Section 10.4.1). We can see the cause of this as coming from the *continuous compouding* used in the definition of the forward rate:

$$B(t,T) = e^{-\int_t^T f(t,u)du},$$

or equivalently

$$B(t,T+\delta)e^{\int_T^{T+\delta} f(t,u)du} = B(t,T).$$

Thus, $f(t,u), T \leq u \leq T+\delta$ can be seen as the interest rate one can lock in at time t for investing on the time interval $[T, T+\delta]$, *compounding continuously.*

A solution to this problem, if we insist on the possibility of positing a log-normal distribution, is to try the *simple compounding* instead. That is, we denote $L_\delta(t,T)$ as the quantity that satisfies

$$B(t,T+\delta)(1+\delta L_\delta(t,T)) = B(t,T).$$

Compare this equation with the one above, you should see that $L_\delta(t,T)$ the interest rate one can lock in at time t for investing on the time interval $[T, T+\delta]$ with *simple compounding: repayment = investment × (1 + duration of investment × interest rate).*

$L_\delta(t,T)$ is called the simple forward LIBOR rate of tenor δ.

1.2 How to construct a portfolio that realize the simple interest rate $L_\delta(t,T)$

Suppose at time $t<T$, we go short one zero-coupon bond and long $B(t,T)/B(t,T+\delta)$ zero-coupon bonds. The value of this portfolio is zero at time t; at time T it requires us to pay out one dollar and at time $T+\delta$ we receive $B(t,T)/B(t,T+\delta)$ dollars. Thus at time t we can lock in a deposit that multiplies to $B(t,T)/B(t,t+\delta)$ over $[T,T+\delta]$ and hence earns the simple interest rate $L_\delta(t,T)$ satisfying

$$1+\delta L_\delta(t,T)=\frac{B(t,T)}{B(t,T+\delta)}$$

Thus

$$L_\delta(t,T)=\frac{1}{\delta}\left[\frac{B(t,T)}{B(t,T+\delta)}-1\right]=\frac{1}{\delta}\frac{B(t,T)-B(t,T+\delta)}{B(t,T+\delta)}.$$

We have immediately that

$$1+\delta L_\delta(T,T)=\frac{1}{B(T,T+\delta)}.$$

Thus $L_\delta(T,T)$ is the simple interest rate available at time T for a deposit over time period $[T,T+\delta]$. This is a financially important quantity, because it is often used for floating rate loans or as a benchmark for interest rate caps and floors.

1.3 Dynamics of $L_\delta(t,T)$

Here is an elementary, but very important observation:

$$\begin{aligned} L_\delta(t,T) &= \frac{1}{\delta}\frac{B(t,T)-B(t,T+\delta)}{B(t,T+\delta)} \\ &= \frac{\frac{1}{\delta}B(t,T)-\frac{1}{\delta}B(t,T+\delta)}{B(t,T+\delta)}. \end{aligned}$$

Thus $L_\delta(t,T)$, for $t\le T$ is the $T+\delta$ forward price of a portfolio that is long $1/\delta$ zero coupon bonds that mature at T and short $1/\delta$ zero coupon bonds that mature at $T+\delta$.

In this section, we will derive the model implied for the forward LIBOR rate by the risk-neutral HJM model. To start out, observe that since

$$\begin{aligned} L_\delta(t,T) &= \frac{1}{\delta}\frac{B(t,T)-B(t,T+\delta)}{B(t,T+\delta)} \\ &= \frac{1}{\delta}\frac{B(t,T)}{B(t,T+\delta)}-\frac{1}{\delta}, \end{aligned}$$

we have

$$dL_\delta(t,T) = \delta^{-1} d[B(t,T)/B(t,T+\delta)].$$

Following the notation of the change of numéraire section, we define

$$B^{T+\delta}(t,T) := B(t,T)/B(t,T+\delta)$$

as the $T+\delta$ forward price of $B(t,T)$.

Observe then, that it is most natural to express the model for $L_\delta(t,T)$ under the $T+\delta$ forward measure $\widetilde{\mathbf{P}}^{T+\delta}$. We know from Theorems 9.2.1 and 9.2.2 in Shreve that because

$$\begin{aligned} dD(t)B(t,T) &= -D(t)B(t,T)\sigma^*(t,T)\, d\widetilde{W}(t) \\ dD(t)B(t,T+\delta) &= -D(t)B(t,T+\delta)\sigma^*(t,T+\delta)\, d\widetilde{W}(t), \end{aligned}$$

we have

$$\begin{aligned} dL_\delta(t,T) &= \frac{1}{\delta} B^{T+\delta}(t,T)[\sigma^*(t,T+\delta) - \sigma^*(t,T)]\, d\widetilde{W}^{T+\delta}(t) \\ &= \frac{1}{\delta}\,[1+\delta L_\delta(t,T)]\,[\sigma^*(t,T+\delta) - \sigma^*(t,T)]\, d\widetilde{W}^{T+\delta}(t) \\ &= L_\delta(t,T)\left\{\frac{1+\delta L_\delta(t,T)}{\delta L_\delta(t,T)}[\sigma^*(t,T+\delta) - \sigma^*(t,T)]\right\} d\widetilde{W}^{T+\delta}(t), \end{aligned} \tag{1}$$

where $\widetilde{W}^{T+\delta}(t) = \widetilde{W}(t) + \int_0^t \sigma^*(u, T+\delta)\, du$ is a Brownian motion under $\widetilde{\mathbf{P}}^{T+\delta}$. From this equation we can easily derive the model for the forward LIBOR rate under the original risk-neutral measure $\widetilde{\mathbf{P}}$, but we will not have need for this.

Remark:

(i) If we denote

$$\gamma(t) := \frac{1+\delta L_\delta(t,T)}{\delta L_\delta(t,T)}[\sigma^*(t,T+\delta) - \sigma^*(t,T)],$$

then it follows that

$$dL_\delta(t,T) = L_\delta(t,T)\gamma(t) d\widetilde{W}^{T+\delta}(t).$$

Thus $L_\delta(t,T)$ has log-normal distribution under $\widetilde{\mathbf{P}}^{T+\delta}$, which is a goal we have set out to achieve. This will help us to derive pricing equation in Black-Scholes style for financial products based on $L_\delta(t,T)$ as discussed in the Sections below.

(ii) Note also that $L_\delta(t,T)$ is a *martingale* under $\widetilde{\mathbf{P}}^{T+\delta}$, a fact which we might also infer from its definition.

2 T-forward models

Previously, we defined a T-forward measure. This is a measure, $\widetilde{\mathbf{P}}^T$, if it exists, under which T-forward prices of all market assets are martingales. Recall that the T-forward price of an asset whose price in dollars is $S(t)$ is $S(t)/B(t,T)$. Now assume we have an HJM model driven by a single Brownian motion, and write it under the risk-neutral measure $\widetilde{\mathbf{P}}$. According to the theory developed in Chapter 9 of Shreve, the T-forward measure is defined by a change of measure from $\widetilde{\mathbf{P}}$ by the Radon-Nikodym derivative,

$$\frac{d\widetilde{\mathbf{P}}^T}{d\widetilde{\mathbf{P}}} = \frac{D(T)}{B(0,T)}. \tag{2}$$

That is, $\widetilde{\mathbf{P}}^T(A) = E[\mathbf{1}_A D(T)]/B(0,T)$, for $A \in \mathcal{F}$. But we know the solution to (**??**) is

$$D(t)B(t,T) = B(0,T)\exp\{-\int_0^t \sigma^*(u,T)\,dW(u) - \frac{1}{2}\int_0^t (\sigma^*)^2(u,T)\,du\}$$

and hence

$$\frac{d\widetilde{\mathbf{P}}^T}{d\widetilde{\mathbf{P}}} = \exp\{-\int_0^T \sigma^*(u,T)\,dW(u) - \frac{1}{2}\int_0^t (\sigma^*)^2(u,T)\,du\}. \tag{3}$$

It follows from Girsanov's theorem that

$$\widetilde{W}^T(t) = \widetilde{W}(t) + \int_0^t \sigma^*(u,T)\,du \tag{4}$$

is a Brownian motion under $\widetilde{\mathbf{P}}^T$, at least for times $t \le T$.

All this is review of section 9.4 in Shreve.

3 Changing between T-forward measures

This section states a formula that will be helpful for understanding forward LIBOR models. Let $0 < T < T'$. Suppose that we have a risk-neutral model for the T' forward prices of a market in which zero-coupon bonds are offered on all maturities. We are not assuming that this has necessarily been derived from an HJM model, just that we have a probability space with a measure $\widetilde{\mathbf{P}}^{T'}$ under which the T'-forward prices of all assets are martingales. Let us denote the T' forward price of an asset whose price in dollars is $S(t)$ by $S^{T'}(t) = S(t)/B(t,T')$. In particular, the T'-forward price of a zero-coupon bond maturing at T, which is

$$B^{T'}(t,T) = \frac{B(t,T)}{B(t,T')}, \quad t \le T,$$

is a martingale under $\widetilde{\mathbf{P}}^{T'}$. The T forward price of an asset whose T' forward price is $S^{T'}(t)$ is

$$S^T(t) = \frac{S(t)}{B(t,T)} = \frac{S(t)/B(t,T')}{B(t,T)/B(t,T')} = \frac{S^{T'}(t)}{B^{T'}(t,T)}.$$

We are interested in finding the $\widetilde{\mathbf{P}}^T$-forward measure that makes prices $S^T(t)$ into martingales. Since we are not starting from an HJM model as in the previous section, we want to derive this in terms of the T'-forward measure. Denote expectation with respect to $\widetilde{\mathbf{P}}^{T'}$ by $\tilde{E}^{T'}$.

Theorem 1. *Define,* $\widetilde{\mathbf{P}}^T$ *by*

$$\widetilde{\mathbf{P}}^T(A) = \frac{B(0,T')}{B(0,T)}\tilde{E}^{T'}[\mathbf{1}_A \frac{1}{B(T,T')}] \tag{5}$$

Then if an asset is such that its T'- forward price is a martingale under $\widetilde{\mathbf{P}}^{T'}$ then its T-forward price is also a martingale under $\widetilde{\mathbf{P}}^T$.

This theorem is a generalization of formula (9.2.7) in Shreve.

Heuristic idea:

The intuitive idea why formula (5) is true is as followed. We want to convert from $\widetilde{\mathbf{P}}^{T'}$ to $\widetilde{\mathbf{P}}^T$. The numéraire associated with $\widetilde{\mathbf{P}}^T$ is $B(t,T)$. The price process of this numéraire under $\widetilde{\mathbf{P}}^{T'}$ is

$$N(t) := \frac{B(t,T)}{B(t,T')}.$$

Thus the change of measure formula states that

$$\begin{aligned}\widetilde{\mathbf{P}}^T(A) &= \tilde{E}^{T'}[\mathbf{1}_A \frac{N(t)}{N(0)}] \\ &= \frac{B(0,T')}{B(0,T)}\tilde{E}^{T'}[\mathbf{1}_A \frac{1}{B(T,T')}].\end{aligned}$$

Compare this with what we did for change of measure from $\widetilde{\mathbf{P}}$ to $\widetilde{\mathbf{P}}^{(N)}$, for example. The numéraire under $\widetilde{\mathbf{P}}^{(N)}$ is clearly $N(t)$. Its "price" under $\widetilde{\mathbf{P}}$ is $D(t)N(t)$. Therefore the change of measure formula is

$$\widetilde{\mathbf{P}}^{(N)}(A) = \widetilde{E}[\mathbf{1}_A \frac{D(t)N(t)}{D(0)N(0)}]$$

Rigorous proof:

The proof is an application of Lemma 5.2.2 in Shreve: Suppose that $Z(t)$ is a positive martingale under a probability measure $\mathbf{P}$ and define

$$\mathbf{P}^Z(A) = E[\mathbf{1}_A Z(T)]/Z(0).$$

Then if $M(t)$ is a martingale under $\mathbf{P}$,

$$\{M(t)/Z(t);\, t \le T\}$$

is a martingale under $\mathbf{P}^Z$. To prove the theorem, simply apply this principle with $\widetilde{\mathbf{P}}$ in place of $\mathbf{P}$ and $B^{T'}(t,T) = B(t,T)/B(t,T')$ in place of $Z(t)$. Note that the definition in (5) is the same as

$$\widetilde{\mathbf{P}}^T(A) = \tilde{E}^{T'}[\mathbf{1}_A B^{T'}(T,T)]/B^{T'}(0,T).$$

Since a T' forward price $S^{T'}(t)$ is a martingale under $\widetilde{\mathbf{P}}^{T'}$, it follows that the T forward price

$$S^T(t) = S^{T'}(t)/B^{T'}(t,T),$$

is a martingale under $\widetilde{\mathbf{P}}^T$ as defined in (5). This completes the proof.

4 Financial products based on forward LIBOR

4.1 Description

The forward LIBOR $L_\delta(t,T)$ is strictly not a financial asset by itself. However, if we think about investing a principal P at time T for the duration $[T, T+\delta]$ to realize the interest payment $P\delta L_\delta(T,T)$ at time $T+\delta$, then we have a product that is very much like a Euro style derivative, with expiry $T+\delta$.

One can also create another product that is in the spirit of the Euro Call option, in this case called an *interest rate cap*. For a constant K positive, we can consider a financial product that pays

$$V_{T+\delta} = \delta P\big(L_\delta(T,T) - K\big)^+$$

at time $T+\delta$. The interpretation is that if we borrow an amount P at time T, we may not want the interest rate $L_\delta(T,T)$ to go beyond K. Therefore to protect ourselves, we would want to get an interest rate cap that would pay us the difference should the interest rate go beyond K.

Moreover, since P and δ are deterministic (we think of them as determined at time 0), for simplicity we can take $P\delta = 1$. Thus, one can discuss the following products:

(i) A contract that pays $L_\delta(T,T)$ at time $T+\delta$. This is called a backset LIBOR on a notional amount of 1.

(ii) A contract that pays $(L_\delta(T,T)-K)^+$ at time $T+\delta$. This is called an *interest rate caplet.*

Clearly the question is what are the risk neutral prices of these products at time 0. We will give the formula for backset LIBOR in this section and give a detailed discussion of interest rate cap and caplet in the next section.

4.2 Risk neutral price of backset LIBOR

Theorem 4.1. *The no arbitrage price at time t of a contract that pays $L_\delta(T,T)$ at time $T+\delta$ is*

$$\begin{aligned} S(t) &= B(t,T+\delta)L_\delta(t,T),\ 0\le t\le T \\ &= B(t,T+\delta)L_\delta(T,T),\ T\le t\le T+\delta. \end{aligned}$$

($S(t)$ is the notation Shreve used in the textbook. Don't confuse it with the stock price).

Proof:

By the risk neutral pricing formula

$$S(t) = \widetilde{E}\Big[e^{-\int_t^{T+\delta} R(u)du}L_\delta(T,T)\Big|\mathcal{F}(t)\Big].$$

If $T\le t$ then $L_\delta(T,T)$ is $\mathcal{F}(t)$ measurable. Therefore

$$S(t) = L_\delta(T,T)\widetilde{E}\Big[e^{-\int_t^{T+\delta} R(u)du}\Big|\mathcal{F}(t)\Big] = B(t,T+\delta)L_\delta(T,T).$$

If $t<T$ then by the change of numéraire pricing formula under $\widetilde{\mathbf{P}}^{T+\delta}$ we have

$$\frac{S(t)}{B(t,T+\delta)} = \widetilde{E}^{T+\delta}\Big[L_\delta(T,T)\Big|\mathcal{F}(t)\Big].$$

But $L_\delta(t,T)$ is a martingale under $\widetilde{\mathbf{P}}^{T+\delta}$ (see equation 1 in Section 1). Therefore,

$$\frac{S(t)}{B(t,T+\delta)} = L_\delta(t,T)$$

and the conclusion follows.

5 Caps and caplets

5.1 Description

We will consider the following type of floating rate bond. It starts at $T_0 = 0$ and pays coupons $C_1, \dots, C_{n+1}$ on principal P at dates $T_1 = \delta$, $T_2 = 2\delta, \dots, T_j = j\delta, \dots, T_{n+1} = (n+1)\delta$. The interest charged over $[T_{j-1}, T_j]$ is the LIBOR rate set at T_{j-1}. So coupon $C_j = \delta P L_\delta(T_{j-1}, T_{j-1})$.

Suppose now that Alice has issued such a bond. An equivalentl interpretation is *she has taken out a floating rate loan.* For convenience, assume the principal is \$1. She can purchase an *interest rate cap* to protect herself against unacceptable increases in the floating rate.

A cap set at strike K and lasting until T_{n+1} will pay her $\delta(L_\delta(T_{j-1}, T_{j-1}) - K)^+$ at each time T_j, $1 \le j \le n+1$. This means that she will never pay more than rate K over any period; the cap will make up the difference between the $\delta L_\delta(T_{j-1}, T_{j-1})$ she owes the bond holder and the maximum δK she wishes to pay. We shall use $\mathrm{Cap}^{\mathrm{m}}(0, n+1)$ to denote the market price of this cap at time $T_0 = 0$.

Consider the derivative which pays the interest rate cap only at time T_j. So it consists of the single payoff $\delta(L_\delta(T_{j-1}, T_{j-1}) - K)^+$ at T_j. This is called a *caplet.* Caplets are not traded as such. However, we can imagine them for the purposes of pricing. Clearly, if $\mathrm{Caplet}_j(0)$ denotes the price of this caplet at time $T_0 = 0$, the total price at $T_0 = 0$ of a cap of maturity T_{n+1} will be

$$\sum_{j=1}^{n+1} \mathrm{Caplet}_j(0).$$

If caps of all maturities are available on the market, we can create a caplet with payoff at T_j by going long one cap maturing at T_j and short one cap maturing at T_{j-1}. Thus the market price of the caplet at T_j is

$$\mathrm{Caplet}_j(0) = \mathrm{Cap}^m(0, j) - \mathrm{Cap}^m(0, j-1).$$

Just as there are interest rate caps, there are also interest floors. By going long a cap and short a floor, one can create also a *collar* that keeps the interest rate one pays between two levels.

Interest rate caps and floors are widely traded and their prices are readily available from the market.

5.2 A remark on the Black-Scholes formula

The pricing formula for the caplet follows the argument of the Black-Scholes formula. The derivation of the Black-Scholes formula is a direct consequence of the following result about normal random variables, which in turn is a consequence of Corollary 1 in the class lecture notes, *Review of Mathematical Finance I.*

Theorem 2. *If Y is a normal random variable with mean 0 and variance ν^2,*

$$E\left[\left(xe^{Y-\nu^2/2}-K\right)^+\right]=xN\left(\frac{\ln(x/K)+\nu^2/2}{\nu}\right)-KN\left(\frac{\ln(x/K)-\nu^2/2}{\nu}\right). \tag{6}$$

To see the connection to the Black-Scholes formula, note that the price at time 0 of a call with strike K is

$$e^{-rT}\tilde{E}\left[\left(xe^{\sigma\widetilde{W}(T)+rT-\frac{1}{2}\sigma^2T}-K\right)^+\right]=e^{-rT}\tilde{E}\left[\left(xe^{rT}e^{\sigma\widetilde{W}(T)-\frac{1}{2}\sigma^2T}-K\right)^+\right].$$

Since $\sigma\widetilde{W}(T)$ is a normal random variable with mean 0 and variance σ^2T, we are exactly in the situation of Theorem 2, and it is easy to derive the Black-Scholes formula from (6).

5.3 Black's caplet model and price formula

The idea behind Black's caplet model and price is to take advantage of Theorem 2 by positing lognormal models where possible. We already saw this strategy in section 9.4 of Shreve, where we assumed T-forward prices for a given T were lognormal. The idea for caplets is similar. Consider the caplet that pays $\delta(L_\delta(T_j,T_j)-K)^+$ at T_{j+1}. We posit that there is a risk-neutral model $\widetilde{\mathbf{P}}^{T_{j+1}}$ under which T_{j+1} forward prices are martingales, that there is a Brownian motion $\widetilde{W}^{T_{j+1}}$ under $\widetilde{\mathbf{P}}^{T_{j+1}}$ and that

$$dL_\delta(t,T_j)=\gamma(t,T_j)L_\delta(t,T_j)\,d\widetilde{W}^{T_{j+1}}, \tag{7}$$

where $\gamma(t,T_j)$ is deterministic. Equivalently,

$$L_\delta(t,T_j)=L_\delta(0,T_j)\exp\left\{\int_0^t\gamma(u,T_j)\,d\widetilde{W}^{T_{j+1}}(u)-\frac{1}{2}\int_0^t\gamma^2(u,T_j)\,du\right\}.$$

For convenience of notation, let

$$\bar{\gamma}^2(T_j)=\frac{1}{T_j}\int_0^{T_j}\gamma^2(u,T_j)\,du.$$

Let $\mathbf{Caplet}_{j+1}(0, \bar{\gamma}(T_j))$ denote the price at $T_0 = 0$ of the caplet maturing at T_{j+1}); (we will see that this price depends only on $\bar{\gamma}(T_j)$, if δ and K are fixed, so the notation is appropriate.) By the risk-neutral pricing formula, the T_{j+1}-forward price of the caplet is

$$\frac{\mathbf{Caplet}_{j+1}(0, \bar{\gamma}(T_j))}{B(0, T_{j+1})} = \delta \tilde{E}^{T_{j+1}} \left[\left(L_\delta(0, T_j) e^{\int_0^{T_j} \gamma(u,T_j)\, d\widetilde{W}^{T_{j+1}}(u) - \frac{1}{2}\int_0^{T_j} \gamma^2(u,T_j)\, du} - K \right)^+ \right].$$

But, since $\gamma(t, T_j)$ is deterministic, $\int_0^{T_j} \gamma(u, T_j)\, d\widetilde{W}^{T_{j+1}}(u)$ is a normal random variable with mean 0 and variance $\int_0^{T_j} \gamma^2(u, T_j)\, du = T_j \bar{\gamma}(T_j)$. Thus from Theorem 2,

$$\begin{aligned} \frac{\mathbf{Caplet}_{j+1}(0, \bar{\gamma}(T_j))}{B(0, T_{j+1})} &= \delta L_\delta(0, T_j) N \left(\frac{\ln \frac{L_\delta(0,T_j)}{K} + \frac{1}{2}\bar{\gamma}^2(T_j) T_j}{\bar{\gamma}(T_j)\sqrt{T_j}} \right) \\ &\quad - \delta K N \left(\frac{\ln \frac{L_\delta(0,T_j)}{K} - \frac{1}{2}\bar{\gamma}^2(T_j) T_j}{\bar{\gamma}(T_j)\sqrt{T_j}} \right) \end{aligned}$$

In this way, we derive *Black's caplet formula*:

$$\begin{aligned} \mathbf{Caplet}_{j+1}(0, \bar{\gamma}(T_j)) &= B(0, T_{j+1}) \Bigg[\delta L_\delta(0, T_j) N \left(\frac{\ln \frac{L_\delta(0,T_j)}{K} + \frac{1}{2}\bar{\gamma}^2(T_j) T_j}{\bar{\gamma}(T_j)\sqrt{T_j}} \right) \\ &\quad - \delta K N \left(\frac{\ln \frac{L_\delta(0,T_j)}{K} - \frac{1}{2}\bar{\gamma}^2(T_j) T_j}{\bar{\gamma}(T_j)\sqrt{T_j}} \right) \Bigg] \end{aligned} \tag{8}$$

The implied spot volatility is the value of $\bar{\gamma}_j$, which, when substituted into Black's caplet formula, give the market value:

$$\mathbf{Caplet}_{j+1}(0, \gamma_j) = \mathrm{Caplet}_{j+1}(0).$$

By finding the implied volatilities and then choosing $\gamma(t, T_j)$ for each j so that

$$\int_0^{T_j} \gamma^2(u, T_j)\, du = T_j \gamma_j,$$

we can fit Black's model to the market for all j.

We emphasize that this model is formulated directly for forward LIBOR and does not assume that one has formulated a prior model, such as an HJM model, for zero-coupon bond prices.